Note that Eq. (16) contains no true secular term with the argument ξ; this is a coincidence that needs not prevail to higher orders.

The solution of (16), subject to the initial conditions for θ_1, is

$$\theta_1 = \frac{K^2R^2\eta \sin 2i}{6} [2 \sin(\psi - \varphi_0) - \sin(\varphi + \psi) + 3 \sin(\varphi - \psi)] + \frac{\delta}{2} \cos\varphi_0 \sin\psi - \frac{\delta}{J\varphi_1} \sin\left(\frac{J\varphi_1\psi}{2}\right) \times \cos\left(\varphi - \frac{J\varphi_1\psi}{2}\right) \quad (17)$$

The last term of Eq. (17) comprises the response to the almost resonant forcing function $\delta \sin\varphi$ and part of the complementary solution. If $\varphi_1 \neq 0$, this term describes a beating oscillation whose amplitude is $O(1/J)$ in violation of the assumed form for the asymptotic expansion of θ. Furthermore, in the limit as $\varphi_1 \to 0$ the term in question tends to $-(\delta/2)\psi \cos\xi$ which is unbounded. This difficulty is easily averted by choosing $\delta = 0$, i.e., $\omega = -K^2R^2 \cos i$. This means that if the node rotates at the just-mentioned rate the representation for θ is bounded to order J in this rotating frame.

When the known functions appearing on the right-hand side of Eq. (8) are substituted, it can be shown that all terms with the argument $\varphi = (1 + J\varphi_1)\xi$ drop out, another occurrence that needs not hold to higher orders. The true secular term with the argument ξ can be eliminated by choosing

$$\varphi_1 = 2K^2R^2 (1 - \tfrac{5}{4} \sin^2 i)$$

and u_1 can be expressed in terms of bounded functions in the form

$$\begin{aligned} u_1 = (K^3R^2/24)\{ &[-(8 + 15\eta + 16\eta^2)\sin^2 i \sin 2\varphi_0] \sin\psi + \\ &[-(72 + 8\eta^2) + (84 + 12\eta^2) \sin^2 i + \\ &(-20 - 27\eta + 4\eta^2) \sin^2 i \cos 2\varphi_0] \cos\psi - \\ &(4 + 6\eta^2) \sin^2 i \cos 2\varphi + 2\eta^2(3 \sin^2 i - 2) \cos 2\psi - \\ &5\eta \sin^2 i \cos(2\varphi + \psi) - \eta^2 \sin^2 i \cos 2(\varphi + \psi) + \\ &3\eta^2 \sin^2 i \cos 2(\varphi - \psi) + 72 + 12\eta^2 - \\ &(84 + 18\eta^2) \sin^2 i + (24 + 32\eta) \sin^2 i \cos 2\varphi_0\} \quad (18) \end{aligned}$$

The just mentioned asymptotic expansions for u and θ are bounded to order J only because Eq. (16) does not contain a term with the argument ξ, and in the differential equation for u_1 the terms with the argument φ canceled out.

In Ref. 3 it is pointed out that the solution to order J^2 for this problem breaks down at the "critical inclination" $i = \sin^{-1} 2/5^{1/2}$ (which corresponds to $\varphi_1 = 0$), and this question has been discussed in the subsequent literature.

The extension of the present method to order J^2 undoubtedly will lead to the same difficulty, since there is no guarantee that terms with the argument ξ as well as terms with the argument φ can be eliminated both in the u_2 and θ_2 equations through the choice of only two additional arbitrary constants ω_2 and φ_2.

It is not surprising that the method of Lindstedt will fail to order J^2 since this method is strictly applicable to the case of *periodic* solutions. In order to derive an asymptotic solution valid for large times for an arbitrary initial value problem of this type it is necessary to use the more general techniques discussed in the literature (cf., Ref. 4).

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Comments on "Prediction and Measurement of Natural Vibrations of Multistage Launch Vehicles"

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In their interesting paper,[1] Alley and Leadbetter present an analytic procedure for finding the natural modes of free-free multistage launch vehicles. A somewhat simpler procedure for finding these free-free modes has been described[2] previously and consists in iterating the matrix equation

$$[R][C][m]\{w_r\} = (1/\omega^2)\{w_r\}$$

where $[C]$ and $[m]$ are the influence coefficient and diagonal mass matrices respectively, and $[R]$ for these free-free missiles is given by

$$[R] = [[1] - (1/M)\{1\}\lfloor m \rfloor - (1/I_y)\{\bar{x}\}\lfloor m\bar{x}\rfloor]$$

All square matrices are of order p, where p is the number of masses. In the forementioned, $\bar{x}$ is distance measured from the center of gravity. It can be shown readily that the elements of $[R]$ just mentioned are identical to those of the $[B]$ matrix of Ref. 1. Furthermore, it was shown in Ref. 2 that the $[C]$ matrix can be any set of influence coefficients for the structure restrained in any way—cantilever, simply-supported, and at any arbitrary points, etc. The restrictions on the influence coefficients mentioned in Ref. 1 are not necessary here, and any convenient set can be used.

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Received March 22, 1963.

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Comment on "The Shock Stand-Off Distance with Stagnation-Point Mass Transfer"

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The Appendix of Ref. 1 contains an analysis of the title subject based on an extension of Lighthill's[2] solution for inviscid hypersonic flow around spheres. The purpose of

Received February 11, 1963.

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this note is to present briefly the more general results previously obtained in Ref. 3 and to point out several inaccurate statements in Ref. 1. The notation is that of Ref. 1.

Lighthill's solution applies for the outer layer between the spherical shock wave and the interface. It should be pointed out that only the second of the three approximations stated in Ref. 1, namely, that of constant density in the flow field between the body and the shock, is required in that analysis. If one then starts with a spherical shock wave and expresses the *general* shock conditions in terms of the shock density ratio, one obtains a problem for which Lighthill's solution is both *exact* and *unique*. The resulting body is a sphere concentric with the shock wave (for further discussion, see Ref. 4). Note that strong shock conditions are not used and that the uniqueness of the solution requires that the shock wave be spherical if the body is assumed to be a sphere. It is unfortunate that Lighthill's original paper is misleading on these two points.

The solution in the inner layer presented in Ref. 1 can be generalized[3] if one relaxes the assumption of injection normal to the body surface [Eq. (A10) of Ref. 1]. The equation to be solved is

$$\frac{1}{r^2}\frac{d^2\Psi}{dr^2} - \frac{2\Psi}{r^4} = D = \frac{\omega}{r \sin\varphi} \tag{1}$$

which is a first integral of Eq. (A12) of Ref. 1. Here D is the constant characterizing the rotationality in the inner layer. The solution of Eq. (1), subject to the conditions at the interface [Eqs. (A8) and (A9) of Ref. 1], is

$$\Psi = \frac{Dr_0{}^4}{30}(3\bar{r}^4 - 5\bar{r}^2 + 2\bar{r}^{-1}) + \frac{\beta_c r_0{}^2}{3}(\bar{r}^2 - \bar{r}^{-1}) \tag{2}$$

where $\beta_c = [dv_\varphi/d \sin\varphi]_c$ and is given from the outer layer solution by Eq. (A14) of Ref. 1. Note that the constant D is unspecified so far.

The solution permits vectored injection of the form $(v_r)_w \sim \cos\varphi$, $(v_\varphi)_w \sim \sin\varphi$, or

$$\tan\alpha = (v_\varphi)_w/(v_r)_w = A \tan\varphi \tag{3}$$

where α is the injection angle and the constant A specifies the degree of vectoring.

Using the boundary conditions at the body surface, one obtains for the rotationality constant the expression

$$D = \frac{5\beta_c}{r_0{}^2}\frac{[2 + \bar{R}_0{}^{-3} - 2A(\bar{R}_0{}^{-3} - 1)]}{[\bar{R}_0{}^{-3} + 5 - 6\bar{R}_0{}^2 - A(2\bar{R}_0{}^{-3} - 5 + 3\bar{R}_0{}^2)]} \tag{4}$$

and for the radial velocity at the body surface the expression

$$\frac{(v_r)_w}{\cos\varphi} = \frac{2\beta_c}{3}(\bar{R}_0{}^{-3} - 1) \times \left\{1 + \frac{(2 + \bar{R}_0{}^{-3})/(\bar{R}_0{}^{-3} - 1) - 2A}{(12\bar{R}_0{}^2 - 10 - 2\bar{R}_0{}^{-3})/(2\bar{R}_0{}^{-3} - 5 + 3\bar{R}_0{}^2) + 2A}\right\} \tag{5}$$

which replaces Eq. (A15) of Ref. 1. Equations (A16) and (A18) of Ref. 1 are modified similarly.

Equation (4) shows that the rotationality depends on the injection process. In fact, if A is chosen to be

$$A = (\tfrac{1}{2})(2 + \bar{R}_0{}^{-3})/(\bar{R}_0{}^{-3} - 1) \tag{6}$$

then $D = 0$, and the inner layer is irrotational. The radial velocity at the body surface then becomes simply

$$(v_r)_w = (\tfrac{2}{3})\beta_c \cos\varphi(\bar{R}_0{}^{-3} - 1) \tag{7}$$

It would appear from physical considerations that, if the injection process were carried out without losses, the inner layer should be irrotational. Results for shock stand-off distance and pressure distribution based on Eqs. (6) and (7) may be found in Ref. 3. The possibility of an irrotational inner layer is in contradiction to the statement in Ref. 1 that this layer must be rotational due to the shear stresses inside the interface.

A final remark is in order concerning the empirical correction for the velocity gradient β_e of Lighthill's solution. The discrepancy between theory and experiment shown in Fig. 6 of Ref. 1 is of the order that one would expect from the constant density approximation. Actually the experimental conditions do not satisfy some of the other assumptions of the analysis. The injection rate is uniform, rather than following a cosine law. The injection region subtends a half-angle of 30°, rather than the complete subsonic region as required for the validity of the analysis. Finally, there is no experimental demonstration that the injection is in fact radial, as assumed in the analysis of Ref. 1. Although the substitution of a Newtonian velocity gradient does give somewhat better agreement with experiment, such an inconsistent procedure also can give incorrect results. As an example, in Fig. A-2 of Ref. 1, the tangential velocity immediately behind the shock wave is shown to be greater than that in front of the shock wave. This is in clear violation of the conservation of tangential momentum across the shock wave.

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Author's Reply to Comment by M. Vinokur

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THE present authors were unaware of the Lockheed report by Vinokur[1] and are grateful for having it called to their attention. The analysis of Vinokur employs the same inviscid model as that of the present authors but is more general in that vectored injection is considered in his analysis.

There are only two disagreements between the authors and Vinokur. The authors' analysis was motivated by the experiments that were reported in Ref. 2. In particular, the porous material through which the coolants were injected was fabricated by a powdered metallurgy process. This process results in a random orientation of particles near the exposed surface; the gas thus is injected in a direction normal to the surface on the mean over a surface area large compared to the pore size, which, for the authors', material is on the order of 65 μ, as noted on p. 818 of Ref. 2. In view of this property of the authors' material, their analysis was restricted to radial injection, and "vectoring" cannot be a source of error between theory and experiment as suggested by Vinokur.

Received January 7, 1963.

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